Homework 3

Instructions: Legibly complete each of the following on lined paper and submit on Gradescope.

$\boxed{*}$ 1. Prove that the Fibonacci number $F_n \in \mathbb{N}$ for all $n \in \mathbb{N}$.

2. Prove that $\displaystyle\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all $n \geq 1$.

3. Prove that $\displaystyle\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$ for all $n \geq 1$.

4. Prove that $F_n \geq (\frac{3}{2})^{n-2}$ for all $n \geq 1$.

$\boxed{*}$ 5. Prove that $n! > 2^n$ for all $n \in \mathbb{Z}_{\geq 4}$.

$\boxed{*}$ 6. Prove that every composite number has a prime factor.

$\boxed{*}$ 7. Prove *Euclid's Lemma*: If p is a prime number and $a, b \in \mathbb{Z}$ satisfy $p \mid ab$, then either $p \mid a$ or $p \mid b$.

8. Prove that $\sqrt[3]{6}$ is irrational.

9. Prove that $\log_2(n)$ is irrational for all odd $n \geq 3$.

10. Let $x, y \in \mathbb{R}$ and assume that x is rational and y is irrational.

$\boxed{*}$ (a) Prove that $x + y$ is irrational.

$\boxed{*}$ (b) Prove that xy is irrational provided $x \neq 0$.

11. Prove that every $n \in \mathbb{Z}_{\geq 2}$ can be expressed in the form $n = 2a + 3b$ for some $a, b \in \mathbb{N}$.

12. Let $a, b, c \in \mathbb{Z}$ be arbitrary with $3 \mid a$, $3 \mid b$, and $c \equiv 2 \pmod 3$. Prove that $ax + by \equiv c \pmod 3$ has no solution.

$\boxed{*}$ 13. Consider the following proposition and its proof.

Proposition 1. *All even natural numbers are equal.*

Proof. We proceed by induction on $n \in \mathbb{N}$ to prove that $0 = 2n$.

Base Case: When $n = 0$, our claim is that $0 = 2 \cdot 0$, which is correct. Hence the base case holds.

Inductive Step: Assume the result holds for some $k \in \mathbb{N}$, i.e., assume $0 = 2k$. Hence $2(k+1) = 2k+2 = 0+2 = 2$ by the inductive hypothesis. Thus we need only show $2 = 0$. Indeed, subtracting 1 from both sides of this equation yields $1 = -1$. Squaring both sides yields $1 = 1$, which is true. Hence $2(k+1) = 2 = 0$, as desired.

We conclude the original statement is true. $\square$

Are you convinced by this argument? Why or why not?

14. We proved the following "Quotient-Remainder Theorem for $\mathbb{N}$" in class.

Proposition 2. *Let $n, d \in \mathbb{N}$ with $d \neq 0$. There exist unique $q, r \in \mathbb{N}$ such that $n = dq + r$ and $0 \leq r < d$.*

Finish the proof of the Quotient-Remainder Theorem for $\mathbb{Z}$ using the proposition above. In other words, prove that for all $n, d \in \mathbb{Z}$ with $d \neq 0$ there exist unique $q, r \in \mathbb{Z}$ such that $n = dq + r$ and $0 \leq r < |d|$.

15. Write a proof of the Fundamental Theorem of Arithmetic using Euclid's Lemma and strong induction.